ON THE FOURTH INTEGER DIMENSION SUBGROUP

BY

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ABSTRACT

An example of a group of class 3 with a non-trivial fourth integer dimension subgroup is given.

According to [1], groups with a trivial $(n+1)$st integer dimension subgroup form a quasi-variety which we shall denote by $\mathfrak{Q}_n$. In this note, we describe a set of quasi-identities and prove that all of them are carried out in $\mathfrak{Q}_3$. After that a group of class 3 not satisfying some of these quasi-identities is constructed. That means that the theorem of Magnus [2], stating that in a free group the nth integer dimension subgroup coincides with the nth term of the lower central series, cannot be extended to arbitrary groups.

We introduce some notation and terminology. For any group G, G_n is the nth term of the lower central series of G, $\mathrm{gp}(S)$ is the subgroup of G generated by the subset S, $mG = \mathrm{gp}\,(g^m/g \in G)$, ZG is the integer group ring of the group G, $\Delta(G)$ is the augmentation ideal of ZG, i.e., the ideal generated by the elements of the form $g-e$ $(g \in G)$, $\delta_n(G) = \mathrm{gp}\,(g/g-e \in \Delta^n(G))$ is the nth integer dimension subgroup of G.

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1. A system of quasi-identities

THEOREM *Let there be given integers* k, c_i, d_{ij} $(1 \leqq i, j \leqq k,\ c_i \geqq 0)$ *and elements* $g_1, \cdots, g_k$ *of a group* G. *Let* $2^{c_i} d_{ij} = b_{ij}$. *Suppose the following conditions*

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(1) $b_{ij} + b_{ji} = 0$ $(1 \leqq i, j \leqq k)$,

(2) *if* $c_i = c_j$ *then* d_{ij} *is even,*

(3) $g_i^{2^{c_i}} \in G_2 (1 \leqq i \leqq k)$,

(4) $\prod_{i=1}^{k} g_i^{b_{ij}} \in (2^{c_j} G_2) G_3$ $(1 \leqq j \leqq k)$,

hold. Then

$$\prod_{i=1}^{k} \prod_{j=i+1}^{k} [g_i, g_j]^{b_{ij}} \in \delta_4(G).$$

REMARK. According to the Theorem the following statement holds in $\mathfrak{Q}_3$:

(S) If for some integers k, c_1, d_{ij} $(1 \leqq i, j \leqq k, c_i \geqq 0)$ and elements $g_1, \cdots, g_k$ of a group G, the conditions (1), (2), (3), (4) hold, then

$$\prod_{i=1}^{k} \prod_{j=i+1}^{k} [g_i, g_j]^{b_{ij}} = e.$$

It is easy to see that (S) is equivalent to a conjunction of an infinite number of quasi-identities and to find their explicit form.

PROOF. We have to show that

$$\prod_{i=1}^{k} \prod_{j=i+1}^{k} [g_i, g_j] b_{ij} - e \in \Delta^4(G).$$

As $g_i^{2^{c_i}} \in G_2$, so $g_i^{2^{c_i}} - e \in \Delta^2(G)$ and

$$2^{c_i}(g_i - e) = g_i^{2^{c_i}} - e - \sum_{m=2}^{2^{c_i}} \binom{2^{c_i}}{m} (g_i - e)^m \in \Delta^2(G).$$

According to (1) and (2), b_{ij} is divisible by at least one of the integers 2^{c_i+1} and 2^{c_j+1} and, respectively, $\binom{b_{ij}}{2}$ is divisible either by 2^{c_i} or by 2^{c_j}.

As

$$g_i^{b_{ij}} - e = \sum_{m=1}^{b_{ij}} \binom{b_{ij}}{m} (g_i - e)^m,$$

so

$$b_{ij}(g_i - e) + \binom{b_{ij}}{2} (g_i - e)^2 - (g_i^{b_{ij}} - e) \in \Delta^3(G),$$

and

$$\sum_{i=1}^{k} \sum_{j=1}^{k} \left\{ b_{ij}(g_i - e) + \binom{b_{ij}}{2} (g_i - e)^2 - (g_i^{b_{ij}} - e) \right\} (g_j - e) \in \Delta^4(G).$$

Using the remarks mentioned above we have

$$\binom{b_{ij}}{2}(g_i - e)^2(g_j - e) \in \Delta^4(G).$$

From (4) follows

$$\sum_{i=1}^{k} (g_i^{b_{ij}} - e) \in 2^{c_j}\, \Delta^2(G) + \Delta^3(G)$$

whence

$$\sum_{i=1}^{k} \sum_{j=1}^{k} (g_i^{b_{ij}} - e)(g_j - e) \in \Delta^4(G)$$

Therefore, we have

$$\sum_{i=1}^{k} \sum_{j=1}^{k} b_{ij}(g_i - e)(g_j - e) \in \Delta^4(G)\,.$$

Using (1) we obtain

$$\sum_{i=1}^{k} \sum_{j=1}^{k} b_{ij}(g_i - e)(g_j - e) = \sum_{i=1}^{k} \sum_{j=i+1}^{k} b_{ij}\{(g_i - e)(g_j - e) - (g_j - e)(g_i - e)\}.$$

Now recall the well-known identity (see, for example, [3]):

$$(g_i - e)(g_j - e) - (g_j - e)(g_i - e) = [g_i, g_j] - e + (g_i - e)([g_i, g_j] - e) + \\ + (g_j - e)([g_i, g_j] - e) + (g_j - e)(g_i - e)([g_i, g_j] - e)\,.$$

As $[g_i, g_j] - e \in \Delta^2(G)$, so

$$(g_j - e)(g_i - e)([g_i, g_j] - e) \in \Delta^4(G)\,,$$

$$2^{c_i}(g_i - e)([g_i, g_j] - e) \in \Delta^4(G),$$

$$b_{ij}(g_j - e)([g_i, g_j] - e) \in \Delta^4(G)\,.$$

Hence, $\displaystyle\sum_{i=1}^{k} \sum_{j=i+1}^{k} b_{ij}([g_i, g_j] - e) \in \Delta^4(G)$

and, correspondingly,

$$\prod_{i=1}^{k} \prod_{j=i+1}^{k} [g_i, g_j]^{b_{ij}} - e \in \Delta^4(G)\,,$$

as required.

2. An example of a group of class 3 with a non-vanishing fourth integer dimension subgroup

Let the group G have generators $a_0, a_1, a_2, a_3, b_1, b_2, b_3, c$ and defining relations

$$b_1^{64} = b_2^{16} = b_3^4 = c^{256} = e,$$

$$[b_2, b_1] = [b_3, b_1] = [b_3, b_2] = [c, b_1] = [c, b_2] = [c, b_3] = e,$$

$$a_0^{64} = b_1^{32}, a_1^{64} = b_2^{-4} b_3^{-2}, a_2^{16} = b_1^4 b_3^{-1}, a_3^4 = b_1^2 b_2,$$

$$[a_1, a_0] = b_1 c^2, [a_2, a_0] = b_2 c^8, [a_3, a_0] = b_3 c^{32},$$

$$[a_2, a_1] = c, [a_3, a_1] = c^2, [a_3, a_2] = c^4,$$

$$[b_1, a_1] = c^4, [b_2, a_2] = c^{16}, [b_3, a_3] = c^{64},$$

$$[b_i, a_j] = e \text{ if } i \neq j,\ [c, a_i] = e \text{ for } i = 0,1,2,3.$$

It is easy to verify that $G_2 = \mathrm{gp}(b_1, b_2, b_3, c)$, $G_3 = \mathrm{gp}(c^4)$, $G_4 = E$. An immediate calculation shows that if we set $k = 3$, $c_1 = 6$, $c_2 = 4$, $c_3 = 2$, $d_{11} = 0, d_{12} = 2, d_{13} = 1, d_{21} = -8, d_{22} = 0, d_{23} = 2, d_{31} = -16, d_{32} = -8$, $d_{33} = 0$, $g_1 = a_1$, $g_2 = a_2$, $g_3 = a_3$, then the conditions (1), (2), (3), (4) of the Theorem hold and therefore

$$[g_1, g_2]^{128}[g_1, g_3]^{64}[g_2, g_3]^{32} = c^{128} \in \delta_4(G).$$

It remains to prove that $c^{128} \neq e$. To do this we can use a result from extension theory. In [4, ch. III, p. 8] a condition is given for the extension of one abelian group by another. For our aims, this condition can be formulated as follows:

Let there be given positive integers $n_0, \cdots, n_r$, an abelian group U, automorphisms $\sigma_0, \cdots, \sigma_r$ and elements $u_0, \cdots, u_r, u_{ij}$ $(0 \leqq i, j \leqq r)$ of U. Let the group H have generators $t_0, \cdots, t_r$ and $\bar{u} (u \in U)$ and defining relations

$$\bar{u}\bar{v} = \overline{uv} \, (u, v \in U),$$

$$[\bar{u}, t_i] = \overline{u^{-1}}\, \overline{u^{\sigma_i}} \, (u \in U, 0 \leqq i \leqq r),$$

$$t_i^{n_i} = \bar{u}_i \ (0 \leqq i \leqq r),$$

$$[t_i, t_j] = \bar{u}_{ij} \ (0 \leqq i, j \leqq r).$$

Let the homomorphism $\lambda: U \to H$ be defined by $u\lambda = \bar{u}$. Then λ is a monomorphism if and only if the following conditions hold:

(a) $\sigma_i \sigma_j = \sigma_j \sigma_i$, $\sigma_i^{n_i} = 1$ $\quad (0 \leqq i, j \leqq r)$,

(b) $u_i^{\sigma_i} = u_i$, $u_i^{\sigma_k} = u_i u_{jk}{}^{1 + \sigma_i + \cdots + \sigma_i{}^{ni-1}}$ $\quad (0 \leqq i, k \leqq r)$

(c) $u_{ii} = e$, $u_{ij} u_{ji} = e$ $\quad (0 \leqq i, j \leqq r)$,

(d) $u_{ij}^{\sigma_k} u_{jk}^{-1} u_{ki}^{\sigma_j} u_{ij}^{-1} u_{jk}^{\sigma_i} u_{ki}^{-1} = e$ $\quad (0 \leqq i, j, k \leqq r)$.

We obtain the group G by assuming that U is an abelian group with generators s_1, s_2, s_3, t and defining relations $s_1^{64} = s_2^{16} = s_3^4 = t^{256} = e$, taking automorphisms $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ of U defined by

$$s_1\sigma_1 = \sigma_1 t^4,\ s_2\sigma_2 = s_2 t^{16},\ s_3\sigma_3 = s_3 t^{64},\ s_i\sigma_j = s_i \text{ if } i \neq j,\ t\sigma_i = t$$

and setting

$$r = 3,\ n_0 = 64,\ n_1 = 64,\ n_2 = 16,\ n_3 = 4,\ t_i = a_i,\ \bar{s}_i = b_i,\ \bar{t} = c,\ u_0 = s_1^{32},$$

$$u_1 = s_2^{-4}s_3^{-2},\ u_2 = s_1^4 s_3^{-1},\ u_3 = s_1^2 s_2,\ u_{10} = s_1 t^2,\ u_{20} = s_2 t^8,\ u_{30} = s_3 t^{32},$$

$$u_{21} = t,\ u_{31} = t^2,\ u_{32} = t^4.$$

It can be immediately verified that the conditions (a), (b), (c), (d) hold and, therefore, according to the criterion mentioned above the monomorphism $\phi : u \to G$ defined by $s_i\phi = b_i$, $t\phi = c$ is a monomorphism. Hence $c^{128} = t^{128}\phi \neq e$, as required.

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